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Analytic Bethe ansatz and functional equations for Lie superalgebra $sl(r+1|s+1)$

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Abstract. From the point of view of the Young superdiagram method, an analytic Bethe ansatz is carried out for Lie super algebra $sl(r+1|s+1)$. For the transfer matrix eigenvalue formulae in dressed-vacuum form, we present some expressions, which are quantum analogues of Jacobi–Trudi and Giambelli formulae for Lie superalgebra $sl(r+1|s+1)$. We also propose transfer-matrix functional relations, which are Hirota bilinear difference equations with some constraints.

1. Introduction

In [KNS1], a class of functional relations, the T-system, was proposed. It is a family of functional relations for a set of commuting transfer matrices of solvable lattice models associated with any quantum affine algebras $U_q(\mathcal{G}_r^{(1)})$. Using the T-system, we can calculate various physical quantities [KNS2] such as the correlation lengths of the vertex models and central charges of RSOS models. The T-system is not only a family of transfer-matrix functional relations but also a two-dimensional Toda field equation on discrete spacetime. And it has beautiful pfaffian and determinant solutions [KOS, KNH, TK] (see also [T]).

In [KS1], an analytic Bethe ansatz [R1] was carried out for fundamental representations of the Yangians $Y(\mathcal{G})$[D], where $\mathcal{G} = B_r$, C_r and D_r. That is, eigenvalue formulae in dressed-vacuum form were proposed for the transfer matrices of solvable vertex models. These formulae are Yangian analogues of the Young tableau for $\mathcal{G}$ and satisfy certain semi-standard-like conditions. It had been proven that they are free of poles under the Bethe ansatz equation. Furthermore, for the $\mathcal{G} = B_r$ case, these formulae were extended to the case of finite dimensional modules labelled by skew Young diagrams $\lambda \subset \mu$ [KOS]. In an analytic Bethe ansatz context, the above-mentioned solutions of the T-system correspond to the eigenvalue formulae of the transfer matrices in dressed-vacuum form labelled by rectangular Young diagrams $\lambda = \phi$, $\mu = (m^a)$ (see also [BR, KLWZ, K, KS2, S2]).

The purpose of this paper is to extend similar analyses to the Lie superalgebra $\mathcal{G} = sl(r+1|s+1)$ [Ka] case (see also [C] for a comprehensible account on Lie superalgebras). Throughout this paper, we frequently use similar notation to that presented in [KS1, KOS, TK]. Studying supersymmetric integrable models is important not only in mathematical physics but also in condensed matter physics (cf [EK, FK, KE, S1, ZB]). For example, the supersymmetric $t-J$ model received much attention in connection with high T_c superconductivity. In the supersymmetric models, the R-matrix satisfies the graded Yang–Baxter equation [KulSk]. The transfer matrix is defined as a *super* trace of the monodromy

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matrix. As a result, extra signs appear in the Bethe ansatz equation and eigenvalue formula of the transfer matrix.

There are several unequivalent choices of simple root system for Lie superalgebra. We treat the so-called distinguished simple root system [Ka] in the main text. We introduce the Young superdiagram [BB1], which is associated with a covariant tensor representation. To be precise, this Young superdiagram is different from the classical one in that it carries a spectral parameter u. In contrast to the ordinary Young diagram, there is no restriction on the number of rows. We define a semi-standard-like tableau on it. Using this tableau, we introduce the function $\mathcal{T}_{\lambda\subset\mu}(u)$ (2.15). This should be a fusion transfer matrix of dressed-vacuum form in the analytic Bethe ansatz. We prove the pole-freeness of $\mathcal{T}^a(u) = \mathcal{T}_{(1^a)}(u)$, a crucial property of the analytic Bethe ansatz. Due to the same mechanism presented in [KOS], the function $\mathcal{T}_{\lambda\subset\mu}(u)$ has a determinant expression whose matrix elements are only the functions associated with Young superdiagrams with shape $\lambda = \phi$; $\mu = (m)$ or (1^a). It can be viewed as a quantum analogue of Jacobi–Trudi and Giambelli formulae for Lie superalgebra $sl(r+1|s+1)$. Then one can easily show that the function $\mathcal{T}_{\lambda\subset\mu}(u)$ is free of poles under the Bethe ansatz equation (2.6*a*). Among the above-mentioned eigenvalue formulae of transfer matrix in dressed-vacuum form associated with rectangular Young superdiagrams, we present a class of transfer-matrix functional relations. It is a special case of Hirota bilinear difference equation [H].

Deguchi and Martin [DM] discussed the spectrum of the fusion model from the point of view of representation theory (see also, [MR]). This present paper will partially give us an elemental account of their result from the point of view of the analytic Bethe ansatz.

The outline of this paper is as follows. In section 2, we execute an analytic Bethe ansatz based upon the Bethe ansatz equation (2.6*a*) associated with the distinguished simple roots. The observation that the Bethe ansatz equation can be expressed by a root system of Lie algebra is traced back to [RW] (see also, [Kul] for the $sl(r+1|s+1)$ case). Moreover, Kuniba *et al* [KOS] conjectured that the left-hand side of the Bethe ansatz equation (2.6*a*) can be written as a ratio of certain 'Drinfeld polynomials' [D]. We introduce the function $\mathcal{T}_{\lambda\subset\mu}(u)$, which should be the transfer matrix whose auxiliary space is a finite-dimensional module of super Yangian $Y(sl(r+1|s+1))$ [N] or quantum affine superalgebra $U_q(sl(r+1|s+1)^{(1)})$ [Y], labelled by a skew Young superdiagram $\lambda \subset \mu$. The origin of the function $\mathcal{T}^1(u)$ goes back to the eigenvalue formula of the transfer matrix of the Perk–Schultz model [PS1, PS2, Sc], which is a multicomponent generalization of the six-vertex model (see also [Kul]). In addition, the function $\mathcal{T}^1(u)$ reduces to the eigenvalue formula of the transfer matrix derived by the algebraic Bethe ansatz (for example, the [FK]: $r = 1, s = 0$ case; [EK]: $r = 0, s = 1$ case; [EKS1, EKS2]: $r = s = 1$ case). In section 3, we propose functional relations, the T-system, associated with the transfer matrices in dressed-vacuum form defined in the previous section. Section 4 is devoted to a summary and discussion. In appendix A, we briefly mention the relation between the fundamental L operator and transfer matrix. In this paper, we treat mainly the expressions related to covariant representations. For contravariant ones, we present several expressions in appendix B. Appendices C and D provide some expressions related to non-distinguished simple roots of $sl(1|2)$. Appendix E explains how to represent the eigenvalue formulae of transfer matrices in dressed-vacuum form $\mathcal{T}_m(u)$ and $\mathcal{T}^a(u)$ in terms of the functions $\mathcal{A}_m(u)$, $\mathcal{A}^a(u)$, $\mathcal{B}_m(u)$ and $\mathcal{B}^a(u)$, which are analogous to the fusion transfer matrices of $U_q(\mathcal{G}^{(1)})$ vertex models ($\mathcal{G} = sl_{r+1}, sl_{s+1}$).

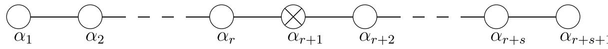

Figure 1. Dynkin diagram for the Lie superalgebra $sl(r+1|s+1)$ corresponding to the distinguished simple roots: open circles denote even roots α_i; crossed circles denote odd roots α_j with $(\alpha_j|\alpha_j) = 0$.

2. Analytic Bethe ansatz

Lie superalgebra [Ka] is a $\mathbb{Z}_2$ graded algebra $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$ with a product $[\,,]$, whose homogeneous elements $a \in \mathcal{G}_\alpha$, $b \in \mathcal{G}_\beta$ ($\alpha, \beta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$) and $c \in \mathcal{G}$ satisfy the following relations.

$$\begin{aligned} &[a, b] \in \mathcal{G}_{\alpha+\beta} \\ &[a, b] = -(-1)^{\alpha\beta}[b, a] \\ &[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]. \end{aligned} \tag{2.1}$$

The set of non-zero roots can be divided into the set of non-zero even roots (bosonic roots) Δ_0' and the set of odd roots (fermionic roots) Δ_1. For the $sl(r+1|s+1)$ case, they read

$$\Delta_0' = \{\epsilon_i - \epsilon_j\} \cup \{\delta_i - \delta_j\} i \neq j \qquad \Delta_1 = \{\pm(\epsilon_i - \delta_j)\} \tag{2.2}$$

where $\epsilon_1, \ldots, \epsilon_{r+1}; \delta_1, \ldots, \delta_{s+1}$ are basis of dual space of the Cartan subalgebra with the bilinear form $(\ |\ )$ such that

$$(\epsilon_i|\epsilon_j) = \delta_{i\,j}, (\epsilon_i|\delta_j) = (\delta_i|\epsilon_j) = 0, (\delta_i|\delta_j) = -\delta_{i\,j}. \tag{2.3}$$

There are several choices of simple root systems reflecting choices of Borel subalgebra. The simplest system of simple roots is the so-called distinguished one [Ka] (see figure 1). Let $\{\alpha_1, \ldots, \alpha_{r+s+1}\}$ be the distinguished simple roots of Lie superalgebra $sl(r+1|s+1)$

$$\begin{aligned} &\alpha_i = \epsilon_i - \epsilon_{i+1} \qquad i = 1, 2, \ldots, r \\ &\alpha_{r+1} = \epsilon_{r+1} - \delta_1 \\ &\alpha_{j+r+1} = \delta_j - \delta_{j+1} \qquad j = 1, 2, \ldots, s \end{aligned} \tag{2.4}$$

and with the grading

$$\deg(\alpha_a) = \begin{cases} 0 & \text{for even root} \\ 1 & \text{for odd root.} \end{cases} \tag{2.5}$$

Especially for the distinguished simple root, we have $\deg(\alpha_a) = \delta_{a,r+1}$.

We consider the following type of the Bethe ansatz equation (cf [Kul, RW, KOS]).

$$-\frac{P_a(u_k^{(a)} + \frac{1}{t_a})}{P_a(u_k^{(a)} - \frac{1}{t_a})} = (-1)^{\deg(\alpha_a)} \prod_{b=1}^{r+s+1} \frac{Q_b(u_k^{(a)} + (\alpha_a|\alpha_b))}{Q_b(u_k^{(a)} - (\alpha_a|\alpha_b))} \tag{2.6a}$$

$$Q_a(u) = \prod_{j=1}^{N_a} [u - u_j^{(a)}] \tag{2.6b}$$

$$P_a(u) = \prod_{j=1}^{N} P_a^{(j)}(u) \tag{2.6c}$$

$$P_a^{(j)}(u) = [u - w_j]^{\delta_{a,1}} \tag{2.6d}$$

where $[u] = (q^u - q^{-u})/(q - q^{-1})$; $N_a \in \mathbb{Z}_{\geqslant 0}$; $u, w_j \in \mathbb{C}$; $a, k \in \mathbb{Z}$ ($1 \leqslant a \leqslant r+s+1$, $1 \leqslant k \leqslant N_a$); $t_a = 1$ for $1 \leqslant a \leqslant r+1$, $t_a = -1$ for $r+2 \leqslant a \leqslant r+s+1$. In this paper, we suppose that q is generic. The left-hand side of the Bethe ansatz equation (2.6*a*) is related to the quantum space. We suppose that it is given by the ratio of some 'Drinfeld polynomials' labelled by skew Young diagrams $\tilde{\lambda} \subset \tilde{\mu}$ (cf [KOS]). For simplicity, we consider only the case $\tilde{\lambda} = \phi, \tilde{\mu} = (1)$. The generalization to the case for any skew Young diagram will be achieved by the empirical procedures mentioned in [KOS]. The factor $(-1)^{\deg(\alpha_a)}$ of the Bethe ansatz equation (2.6*a*) appears so as to make the transfer matrix to be a *super* trace of the monodromy matrix. We define the sets

$$J = \{1, 2, \ldots, r+s+2\} \qquad J_+ = \{1, 2, \ldots, r+1\}$$
$$J_- = \{r+2, r+3, \ldots, r+s+2\} \tag{2.7}$$

with the total order

$$1 \prec 2 \prec \cdots \prec r+s+2 \tag{2.8}$$

and with the grading

$$p(a) = \begin{cases} 0 & a \in J_+ \\ 1 & \text{for } a \in J_-. \end{cases} \tag{2.9}$$

For $a \in J$, set

$$z(a;u) = \psi_a(u) \frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \qquad \text{for } a \in J_+$$
$$z(a;u) = \psi_a(u) \frac{Q_{a-1}(u+2r-a+1)Q_a(u+2r-a+4)}{Q_{a-1}(u+2r-a+3)Q_a(u+2r-a+2)} \qquad \text{for } a \in J_- \tag{2.10}$$

where $Q_0(u) = 1$, $Q_{r+s+2}(u) = 1$ and

$$\psi_a(u) = \begin{cases} P_1(u+2) & \text{for } a = 1 \\ P_1(u) & \text{for } a \in J - \{1\}. \end{cases} \tag{2.11}$$

In this paper, we often express the function $z(a;u)$ as the box $\boxed{a}_u$, whose spectral parameter u will often be abbreviated. Under the Bethe ansatz equation, we have

$$\mathrm{Res}_{u=-b+u_k^{(b)}}(z(b;u) + z(b+1;u)) = 0 \qquad 1 \leqslant b \leqslant r \tag{2.12a}$$
$$\mathrm{Res}_{u=-r-1+u_k^{(r+1)}}(z(r+1;u) - z(r+2;u)) = 0 \tag{2.12b}$$
$$\mathrm{Res}_{u=-2r-2+b+u_k^{(b)}}(z(b;u) + z(b+1;u)) = 0 \qquad r+2 \leqslant b \leqslant r+s+1. \tag{2.12c}$$

We will use the functions $\mathcal{T}^a(u)$ and $\mathcal{T}_m(u)$ ($a \in \mathbb{Z}$; $m \in \mathbb{Z}$; $u \in \mathbb{C}$) determined by the following generating series

$$(1+z(r+s+2;u)X)^{-1} \cdots (1+z(r+2;u)X)^{-1}(1+z(r+1;u)X) \cdots (1+z(1;u)X)$$
$$= \sum_{a=-\infty}^{\infty} \mathcal{F}^a(u+a-1)\mathcal{T}^a(u+a-1)X^a \tag{2.13a}$$

$$\mathcal{F}^a(u) = \begin{cases} \displaystyle\prod_{j=1}^{a-1} P_1(u-2j+a-1) & \text{for } a \geqslant 2 \\ 1 & \text{for } a = 1 \\ \dfrac{1}{P_1(u-1)} & \text{for } a = 0 \\ 0 & \text{for } a \leqslant -1 \end{cases} \tag{2.13b}$$

$$(1-z(1;u)X)^{-1} \cdots (1-z(r+1;u)X)^{-1}(1-z(r+2;u)X) \cdots (1-z(r+s+2;u)X)$$

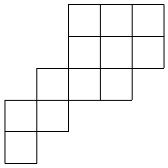

Figure 2. Young superdiagram with shape $\lambda \subset \mu$: $\lambda = (2, 2, 1, 0, 0)$, $\mu = (5, 5, 4, 2, 1)$.

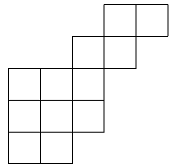

Figure 3. Young superdiagram with shape $\lambda' \subset \mu'$: $\lambda' = (3, 2, 0, 0, 0)$, $\mu' = (5, 4, 3, 3, 2)$.

$$= \sum_{m=-\infty}^{\infty} \mathcal{T}_m(u+m-1)X^m \tag{2.13c}$$

where X is a shift operator $X = \mathrm{e}^{2\partial_u}$. In particular, we have $\mathcal{T}^0(u) = P_1(u-1)$; $\mathcal{T}_0(u) = 1$; $\mathcal{T}^a(u) = 0$ for $a < 0$; $\mathcal{T}_m(u) = 0$ for $m < 0$.

We remark that the origin of the function $\mathcal{T}^1(u)$ and the Bethe ansatz equation (2.6a) traces back to the eigenvalue formula of the transfer matrix and the Bethe ansatz equation of the Perk–Schultz model [Sc] except the vacuum part, some gauge factors and extra signs after some redefinition. (See also [Kul]).

Let $\lambda \subset \mu$ be a skew Young superdiagram labelled by the sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i \geqslant \lambda_i$: $i = 1, 2, \ldots$; $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$; $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ be the conjugate of λ (see figures 2 and 3). On this skew Young superdiagram $\lambda \subset \mu$, we assign coordinates $(i, j) \in \mathbb{Z}^2$ such that the row index i increases as we go downwards and the column index j increases as we go from left to right and that $(1, 1)$ is on the top left corner of μ. We define an admissible tableau b on the skew Young superdiagram $\lambda \subset \mu$ as a set of elements $b(i, j) \in J$ labelled by the coordinates (i, j) mentioned above, obeying the following rule (admissibility conditions).

(i) For any elements of J_+,

$$b(i, j) \preceq b(i, j+1) \qquad b(i, j) \prec b(i+1, j) \tag{2.14a}$$

(ii) for any elements of J_-,

$$b(i, j) \prec b(i, j+1) \qquad b(i, j) \preceq b(i+1, j) \tag{2.14b}$$

(iii) and for any elements of J,

$$b(i, j) \preceq b(i, j+1) \qquad b(i, j) \preceq b(i+1, j). \tag{2.14c}$$

Let $B(\lambda \subset \mu)$ be the set of admissible tableau on $\lambda \subset \mu$.

For any skew-Young superdiagram $\lambda \subset \mu$, define the function $\mathcal{T}_{\lambda\subset\mu}(u)$ as follows

$$\mathcal{T}_{\lambda\subset\mu}(u) = \frac{1}{\mathcal{F}_{\lambda\subset\mu}(u)} \sum_{b \in B(\lambda\subset\mu)} \prod_{(i,j)\in(\lambda\subset\mu)} (-1)^{p(b(i,j))} z(b(i, j); u - \mu_1 + \mu'_1 - 2i + 2j) \tag{2.15}$$

where the product is taken over the coordinates (i, j) on $\lambda \subset \mu$ and

$$\mathcal{F}_{\lambda\subset\mu}(u) = \prod_{j=1}^{\mu_1} \mathcal{F}^{\mu'_j - \lambda'_j}(u + \mu'_1 - \mu_1 - \mu'_j - \lambda'_j + 2j - 1). \tag{2.16}$$

In particular, for an empty diagram ϕ, set $\mathcal{T}_\phi(u) = \mathcal{F}_\phi(u) = 1$. The following relations should be valid for the same reason mentioned in [KOS], that is, they will be verified by induction on μ_1 or μ'_1.

$$\mathcal{T}_{\lambda\subset\mu}(u) = \det_{1\leqslant i,j\leqslant\mu_1}(\mathcal{T}^{\mu'_i-\lambda'_j-i+j}(u-\mu_1+\mu'_1-\mu'_i-\lambda'_j+i+j-1)) \tag{2.17a}$$

$$= \det_{1\leqslant i,j\leqslant\mu'_1}(\mathcal{T}_{\mu_j-\lambda_i+i-j}(u-\mu_1+\mu'_1+\mu_j+\lambda_i-i-j+1)). \tag{2.17b}$$

For example, for the $\lambda=\phi$, $\mu=(2^2)$, $r=1$, $s=0$ case, we have

$$\begin{aligned}\mathcal{T}_{(2^2)}(u) &= \frac{1}{\mathcal{F}_{(2^2)}(u)}\left(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline 2&3\\\hline\end{array}\right)\\
&= P_1(u+2)P_1(u+4)\frac{Q_2(u-2)}{Q_2(u+2)} - P_1(u+2)P_1(u+4)\frac{Q_1(u+1)Q_2(u-2)}{Q_1(u+3)Q_2(u+2)}\\
&\quad -P_1(u+2)^2\frac{Q_1(u+5)Q_2(u-2)}{Q_1(u+3)Q_2(u+4)} + P_1(u+2)^2\frac{Q_2(u-2)}{Q_2(u+4)}\\
&= \begin{vmatrix}\mathcal{T}^2(u-1) & \mathcal{T}^3(u)\\ \mathcal{T}^1(u) & \mathcal{T}^2(u+1)\end{vmatrix}\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}\mathcal{T}^1(u) &= \boxed{1}+\boxed{2}-\boxed{3}\\
&= P_1(u+2)\frac{Q_1(u-1)}{Q_1(u+1)} + P_1(u)\frac{Q_1(u+3)Q_2(u)}{Q_1(u+1)Q_2(u+2)} - P_1(u)\frac{Q_2(u)}{Q_2(u+2)}\end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathcal{T}^2(u) &= \frac{1}{\mathcal{F}^2(u)}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}-\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}-\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}\right)\\
&= P_1(u+3)\frac{Q_2(u-1)}{Q_2(u+1)} - P_1(u+3)\frac{Q_1(u)Q_2(u-1)}{Q_1(u+2)Q_2(u+1)}\\
&\quad -P_1(u+1)\frac{Q_1(u+4)Q_2(u-1)}{Q_1(u+2)Q_2(u+3)} + P_1(u+1)\frac{Q_2(u-1)}{Q_2(u+3)}\end{aligned} \tag{2.20}$$

$$\begin{aligned}\mathcal{T}^3(u) &= \frac{1}{\mathcal{F}^3(u)}\left(-\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 3\\\hline\end{array}-\begin{array}{|c|}\hline 3\\\hline 3\\\hline 3\\\hline\end{array}\right)\\
&= -P_1(u+4)\frac{Q_2(u-2)}{Q_2(u+2)} + P_1(u+4)\frac{Q_1(u+1)Q_2(u-2)}{Q_1(u+3)Q_2(u+2)}\\
&\quad +P_1(u+2)\frac{Q_1(u+5)Q_2(u-2)}{Q_1(u+3)Q_2(u+4)} - P_1(u+2)\frac{Q_2(u-2)}{Q_2(u+4)}.\end{aligned} \tag{2.21}$$

Remark 1. If we drop the u dependence of (2.17a) and (2.17b), they reduce to classical Jacobi–Trudi and Giambelli formulae for $sl(r+1|s+1)$ [BB1, PT], which bring us classical (super) characters.

Remark 2. In the case $\lambda=\phi$ and $s=-1$, (2.17a) and (2.17b) correspond to the quantum analogue of Jacobi–Trudi and Giambelli formulae for sl_{r+1} [BR].

Remark 3. Equations (2.17a) and (2.17b) have the same form as the quantum Jacobi–Trudi and Giambelli formulae for $U_q(B_n^{(1)})$ in [KOS], but the function $\mathcal{T}^a(u)$ is quite different.

The following theorem is essential in analytic Bethe ansatz, which can be proved along the similar line of the proof of [KS1, theorem 3.3.1].

Theorem 2.1. For any integer a, the function $\mathcal{T}^a(u)$ is free of poles under the condition that the Bethe ansatz equation (2.6a) is valid.

At first, we present a lemma which is necessary for the proof of theorem 2.1. Lemma 2.2 is a $sl(r+1|s+1)$ version of [KS1, lemma 3.3.2] and follows straightforwardly from the definitions of $z(a;u)$ (2.10).

Lemma 2.2. For any $b \in J_+ - \{r+1\}$, the function

$$\begin{array}{|c|}\hline b \\ \hline b+1 \\ \hline\end{array}\begin{array}{l} {\scriptstyle u} \\ {\scriptstyle u-2}\end{array} \tag{2.22}$$

does not contain the function Q_b (2.6*b*).

Proof of theorem 2.1. For simplicity, we assume that the vacuum parts are formally trivial, that is, the left-hand side of the Bethe ansatz equation (2.6*a*) is constantly -1. We prove that $\mathcal{T}^a(u)$ is free of the colour b pole, namely, $\mathrm{Res}_{u=u_k^{(b)}+\cdots}\mathcal{T}^a(u) = 0$ for any $b \in J - \{r+s+2\}$ under the condition that the Bethe ansatz equation (2.6*a*) is valid. The function $z(c;u) = \boxed{c}_{\,u}$ with $c \in J$ has the colour b pole only for $c = b$ or $b+1$, so we shall trace only $\boxed{b}$ or $\boxed{b+1}$. Denote S_k the partial sum of $\mathcal{T}^a(u)$, which contains k boxes among $\boxed{b}$ or $\boxed{b+1}$. Apparently, S_0 does not have a colour b pole. This is also the case with S_2 for $b \in J_+ - \{r+1\}$ since the admissible tableau have the same subdiagrams as in (2.22) and thus do not involve Q_b by lemma 2.2. Now we examine S_1 which is the summation of the tableau of the form

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline \zeta \\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline \xi \\ \hline b+1 \\ \hline \zeta \\ \hline\end{array} \tag{2.23}$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-1$ and do not involve $\boxed{b}$ and $\boxed{b+1}$. Thanks to relations (2.12*a*)–(2.12*c*), colour b residues in these tableau (2.23) cancel each other under the Bethe ansatz equation (2.6*a*). We deal with S_k only for $3 \leqslant k \leqslant a$ and $k = 2$ with $b \in J_- \cup \{r+1\} - \{r+s+2\}$ from now on. In this case, only the case for $b \in \{r+1\} \bigcup J_- - \{r+s+2\}$ should be considered because, in the case for $b \in J_+ - \{r+1\}$, $\boxed{b}$ or $\boxed{b+1}$ appear at most twice in one column.

The case $b = r+1$: $S_k(k \geqslant 2)$ is the summation of the tableau of the form

$$\begin{array}{|c|}\hline \xi \\ \hline r+1 \\ \hline r+2 \\ \hline \vdots \\ \hline r+2 \\ \hline \zeta \\ \hline\end{array}\begin{array}{l} \\ {\scriptstyle v} \\ {\scriptstyle v-2} \\ \\ {\scriptstyle v-2k+2} \\ \\ \end{array} = \frac{Q_{r+1}(v+r-2k+1)Q_{r+2}(v+r)}{Q_{r+1}(v+r+1)Q_{r+2}(v+r+2)}X_3 \tag{2.24}$$

and

$$\begin{array}{|c|}\hline \xi \\ \hline r+2 \\ \hline r+2 \\ \hline \vdots \\ \hline r+2 \\ \hline \zeta \\ \hline\end{array}\begin{array}{l} \\ {\scriptstyle v} \\ {\scriptstyle v-2} \\ \\ {\scriptstyle v-2k+2} \\ \\ \end{array} = \frac{Q_{r+1}(v+r-2k+1)Q_{r}(v+r)}{Q_{r+1}(v+r+1)Q_{r}(v+r+2)}X_3 \tag{2.25}$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-k$, which do not contain $\boxed{r+1}$ and $\boxed{r+2}$; $v = u + h_1$: h_1 is some shift parameter; the function X_3 does not contain the function Q_{r+1}. Obviously, colour $b = r+1$ residues in (2.24) and (2.25) cancel each other under the Bethe ansatz equation (2.6a).

The case $b \in J_- - \{r+s+2\}$: $S_k (k \geqslant 2)$ is the summation of the tableau of the form

$$
f(k, n, \xi, \zeta, u) := \begin{array}{|c|l}
\cline{1-1} \xi & \\
\cline{1-1} b & {\scriptstyle v} \\
\cline{1-1} \vdots & \\
\cline{1-1} b & {\scriptstyle v-2n+2} \\
\cline{1-1} b+1 & {\scriptstyle v-2n} \\
\cline{1-1} \vdots & \\
\cline{1-1} b+1 & {\scriptstyle v-2k+2} \\
\cline{1-1} \zeta & \\
\cline{1-1}
\end{array}
$$

$$
\begin{aligned}
&= \frac{Q_{b-1}(v+2r+3-2n-b)Q_b(v+2r+4-b)}{Q_{b-1}(v+2r+3-b)Q_b(v+2r+4-2n-b)} \\
&\quad \times \frac{Q_b(v+2r+2-2k-b)Q_{b+1}(v+2r+3-2n-b)}{Q_b(v+2r+2-2n-b)Q_{b+1}(v+2r+3-2k-b)} X_4 \qquad 0 \leqslant n \leqslant k
\end{aligned}
\tag{2.26}
$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-k$, which do not contain $\boxed{b}$ and $\boxed{b+1}$; $v = u + h_2$: h_2 is some shift parameter and is independent of n; the function X_4 does not have a colour b pole and is independent of n. $f(k, n, \xi, \zeta, u)$ has colour b poles at $u = -h_2 - 2r - 2 + b + 2n + u_p^{(b)}$ and $u = -h_2 - 2r - 4 + b + 2n + u_p^{(b)}$ for $1 \leqslant n \leqslant k-1$; at $u = -h_2 - 2r - 2 + b + u_p^{(b)}$ for $n = 0$ and at $u = -h_2 - 2r - 4 + b + 2k + u_p^{(b)}$ for $n = k$. Evidently, colour b residue at $u = -h_2 - 2r - 2 + b + 2n + u_p^{(b)}$ in $f(k, n, \xi, \zeta, u)$ and $f(k, n+1, \xi, \zeta, u)$ cancel each other under the Bethe ansatz equation (2.6a). Thus, under the Bethe ansatz equation (2.6a), $\sum_{n=0}^{k} f(k, n, \xi, \zeta, u)$ is free of colour b poles, so is S_k.□

Applying theorem 2.1 to (2.17a), one can show that $\mathcal{T}_{\lambda\subset\mu}(u)$ is free of poles under the Bethe ansatz equation (2.6a). The function $\mathcal{T}_{\lambda\subset\mu}(u)$ should express the eigenvalue of the transfer matrix whose auxiliary space $W_{\lambda\subset\mu}(u)$ is labelled by the skew-Young superdiagram with shape $\lambda \subset \mu$. We assume that $W_{\lambda\subset\mu}(u)$ is a finite-dimensional module of the super Yangian $Y(sl(r+1|s+1))$ [N] (or quantum super affine algebra $U_q(sl(r+1|s+1)^{(1)})$ [Y] in the trigonometric case). On the other hand, for the $\lambda = \phi$ case, the highest weight representation of Lie superalgebra $sl(r+1|s+1)$, which is a classical counterpart of $W_\mu(u)$, is characterized by the highest weight whose Kac–Dynkin labels $a_1, a_2, \ldots, a_{r+s+1}$ [BMR] are given as follows:

$$
\begin{aligned}
&a_j = \mu_j - \mu_{j+1} \qquad \text{for } 1 \leqslant j \leqslant r \\
&a_{r+1} = \mu_{r+1} + \eta_1 \\
&a_{j+r+1} = \eta_j - \eta_{j+1} \qquad \text{for } 1 \leqslant j \leqslant s
\end{aligned}
\tag{2.27}
$$

where $\eta_j = \max\{\mu'_j - r - 1, 0\}$; $\mu_{r+2} \leqslant s+1$ for the covariant case. One can read the relations (2.27) from the 'top term' [KS1, KOS] in (2.15) for large q^u (see figure 4). The 'top term' in (2.15) is the term labelled by the tableau b such that

$$
b(i, j) = \begin{cases} i & \text{for } 1 \leqslant j \leqslant \mu_i \text{ and } 1 \leqslant i \leqslant r+1 \\ r+j+1 & \text{for } 1 \leqslant j \leqslant \mu_i \text{ and } r+2 \leqslant i \leqslant \mu'_1. \end{cases}
\tag{2.28}
$$

1	1	1	1	1
2	2	2	2	
3	3	3		
4	5			
4	5			
4				

Figure 4. Young supertableau corresponding to the top term for $sl(3|2)$; $\lambda \subset \mu$: $\lambda = \phi$, $\mu = (5, 4, 3, 2, 2, 1)$.

Then, for large q^u, we have

$$\begin{aligned}
\prod_{(i,j)\in\mu} & (-1)^{p(b(i,j))} z(b(i,j); u + \mu'_1 - \mu_1 - 2i + 2j) \\
&= (-1)^{\sum_{i=r+2}^{\mu'_1} \mu_i} \left\{ \prod_{i=1}^{r+1} \prod_{j=1}^{\mu_i} z(i; u + \mu'_1 - \mu_1 - 2i + 2j) \right\} \\
&\quad \times \left\{ \prod_{j=1}^{\mu_{r+2}} \prod_{i=r+2}^{\mu'_j} z(r + j + 1; u + \mu'_1 - \mu_1 - 2i + 2j) \right\} \\
&\approx (-1)^{\sum_{i=r+2}^{\mu'_1} \mu_i} q^{-2\sum N_b a_b t_b}.
\end{aligned} \tag{2.29}$$

Here we omit the vacuum part ψ_a. The 'top term' is considered to be related with the 'highest weight vector'. See [KS1, KOS], for more details.

3. Functional equations

Consider the following Jacobi identity:

$$D\begin{bmatrix} b \\ b \end{bmatrix} D\begin{bmatrix} c \\ c \end{bmatrix} - D\begin{bmatrix} b \\ c \end{bmatrix} D\begin{bmatrix} c \\ b \end{bmatrix} = D\begin{bmatrix} b & c \\ b & c \end{bmatrix} D \qquad b \neq c \tag{3.1}$$

where D is the determinant of a matrix and $D\begin{bmatrix} a_1 & a_2 & \dots \\ b_1 & b_2 & \dots \end{bmatrix}$ is its minor removing a_α's rows and b_β's columns. Set $\lambda = \phi$, $\mu = (m^a)$ in (2.17*a*). From relation (3.1), we have

$$\mathcal{T}^a_m(u-1)\mathcal{T}^a_m(u+1) = \mathcal{T}^a_{m+1}(u)\mathcal{T}^a_{m-1}(u) + g^a_m(u)\mathcal{T}^{a-1}_m(u)\mathcal{T}^{a+1}_m(u) \tag{3.2}$$

where $a, m \geqslant 1$; $\mathcal{T}^a_m(u) = \mathcal{T}_{(m^a)}(u)$: $a, m \geqslant 1$; $\mathcal{T}^0_m(u) = 1$: $m \geqslant 0$; $\mathcal{T}^a_0(u) = 1$: $a \geqslant 0$; $g^1_m(u) = \prod_{j=1}^m P_1(u - m + 2j - 2)$: $m \geqslant 1$; $g^a_m(u) = 1$: $a \geqslant 2$ and $m \geqslant 0$, or $a = 1$ and $m = 0$. Note that the following relation holds:

$$g^a_m(u+1)g^a_m(u-1) = g^a_{m+1}(u)g^a_{m-1}(u) \qquad \text{for } a, m \geqslant 1. \tag{3.3}$$

The functional equation (3.2) is a special case of the Hirota bilinear difference equation [H]. In addition, there are some restrictions on it, which we consider below.

Theorem 3.1. $\mathcal{T}_{\lambda\subset\mu}(u) = 0$ if $\lambda \subset \mu$ contains a rectangular subdiagram with $r+2$ rows and $s+2$ columns (see [DM, MR]).

Proof. We assume the coordinate of the top-left corner of this subdiagram is (i_1, j_1). Consider the tableau b on this Young superdiagram $\lambda \subset \mu$. Fill the first column of this subdiagram from the top to the bottom by the elements of $b(i, j_1) \in J$: $i_1 \leqslant i \leqslant i_1 + r + 1$, so as to meet the admissibility conditions (i)–(iii). We find $b(i_1+r+1, j_1) \in J_-$. Then we have $r+2 \preceq b(i_1+r+1, j_1) \prec b(i_1+r+1, j_1+1) \prec \cdots \prec b(i_1+r+1, j_1+s+1)$. This contradicts the condition $b(i_1+r+1, j_1+s+1) \preceq r+s+2$. □

As a corollary, we have

$$\mathcal{T}^a_m(u) = 0 \qquad \text{for } a \geqslant r+2 \text{ and } m \geqslant s+2. \tag{3.4}$$

Consider the admissible tableau on the Young superdiagram with shape (m^{r+1}). From the admissibility conditions (i)–(iii), only such tableau as $b(i,j) = i$ for $1 \leqslant i \leqslant r+1$ and $1 \leqslant j \leqslant m-s-1$ are admissible. Then we have,

$$\begin{aligned}
\mathcal{T}^{r+1}_m(u) &= \mathcal{T}_{(m^{r+1})}(u) \\
&= \frac{1}{\mathcal{F}_{(m^{r+1})}(u)} \sum_{b \in B(m^{r+1})} \prod_{(i,j)\in(m^{r+1})} (-1)^{p(b(i,j))} z(b(i,j); u+r+1-m-2i+2j) \\
&= \frac{1}{\mathcal{F}_{(m^{r+1})}(u)} \prod_{i=1}^{r+1} \prod_{j=1}^{m-s-1} (-1)^{p(i)} z(i; u+r+1-m-2i+2j) \\
&\quad \times \sum_{b \in B((s+1)^{r+1})} \prod_{i=1}^{r+1} \prod_{j=m-s}^{m} (-1)^{p(b(i,j))} z(b(i,j); u+r+1-m-2i+2j) \\
&= \mathcal{F}^{m-s}(u+r-s+2) \frac{Q_{r+1}(u-m)}{Q_{r+1}(u+m-2s-2)} \\
&\quad \times \mathcal{T}^{r+1}_{s+1}(u+m-s-1) \qquad m \geqslant s+1.
\end{aligned} \tag{3.5a}$$

Similarly, we have

$$\mathcal{T}^a_{s+1}(u) = (-1)^{(s+1)(a-r-1)} \frac{Q_{r+1}(u-a-s+r)}{Q_{r+1}(u+a-s-r-2)} \mathcal{T}^{r+1}_{s+1}(u+a-r-1) \qquad a \geqslant r+1. \tag{3.5b}$$

From relations (3.5*a*) and (3.5*b*), we obtain the following theorem.

Theorem 3.2. For $a \geqslant 1$ and $r \geqslant 0$, the following relation is valid.

$$\mathcal{T}^{r+1}_{a+s}(u) = (-1)^{(s+1)(a-1)} \mathcal{F}^a(u+r-s+2) \mathcal{T}^{r+a}_{s+1}(u). \tag{3.6}$$

Applying relation (3.4) to (3.2), we obtain

$$\mathcal{T}^{r+1}_m(u-1)\mathcal{T}^{r+1}_m(u+1) = \mathcal{T}^{r+1}_{m+1}(u)\mathcal{T}^{r+1}_{m-1}(u) \qquad m \geqslant s+2 \tag{3.7a}$$

$$\mathcal{T}^a_{s+1}(u-1)\mathcal{T}^a_{s+1}(u+1) = g^a_{s+1}(u)\mathcal{T}^{a-1}_{s+1}(u)\mathcal{T}^{a+1}_{s+1}(u) \qquad a \geqslant r+2. \tag{3.7b}$$

Thanks to theorem 3.2, (3.7*a*) is equivalent to (3.7*b*). From theorem 3.2, we also have

$$\mathcal{T}^{r+1}_{s+1}(u-1)\mathcal{T}^{r+1}_{s+1}(u+1) = \mathcal{T}^{r+1}_{s+2}(u) \left(\mathcal{T}^{r+1}_s(u) + (-1)^{s+1} \frac{\mathcal{T}^r_{s+1}(u)}{\mathcal{F}^2(u+r-s+2)} \right). \tag{3.8}$$

Remark. In the relation (3.5*a*), we assume that the parameter m takes only integer values. However, there is a possibility of m taking non-integer values, except some 'singular point', for example, on which the right-hand side of (3.5*a*) contains constant terms, by 'analytic continuation'. We can easily observe this fact from the right-hand side of (3.5*a*) as long as the normalization factor $\mathcal{F}^{m-s}(u)$ is disregarded. This seems to correspond to the fact that $(r+1)$th Kac–Dynkin label (2.27) a_{r+1} can take a non-integer value [Ka]. Furthermore, these circumstances seem to be connected with the lattice models based upon the solution of the graded Young–Baxter equation, which depends on non-additive continuous parameters (see for example [M, PF]).

4. Summary and discussion

In this paper, we have executed analytic Bethe ansatz for Lie superalgebra $sl(r+1|s+1)$. Pole-freeness of the eigenvalue formula of the transfer matrix in dressed-vacuum form was shown for a wide class of finite dimensional representations labelled by skew Young superdiagrams. A functional relation has been given especially for the eigenvalue formulae of transfer matrices in dressed-vacuum form labelled by rectangular Young superdiagrams, which is a special case of the Hirota bilinear difference equation with some restrictive relations.

It should be emphasized that our method presented in this paper is also applicable even if such factors like extra sign (different from that of (2.6*a*)), gauge factor, etc appear in the Bethe ansatz equation (2.6*a*). This is because such factors do not affect the analytical property of the right-hand side of the Bethe ansatz equation (2.6*a*).

It would be an interesting problem to extend similar analyses to mixed representation cases [BB2]. So far we have only found several determinant representations of mixed tableau. The simplest one is given as follows.

$$\sum_{(a,b)\in X}(-1)^{p(a)+p(b)}\dot{z}(a;u+s)z(b;u+r)=\begin{vmatrix}\dot{\mathcal{T}}^1(u+s) & 1\\ 1 & \mathcal{T}^1(u+r)\end{vmatrix} \qquad (4.1)$$

where $X=\{(a,b):a\in\dot{J};b\in J;(a,b)\neq(-1,1)\}$ for $sl(r+1|s+1):r\neq s$; $\dot{\mathcal{T}}^1(u)$ and $\dot{J}$ are the expressions related to contravariant representations (see appendix B). Here we assume that the vacuum parts are formally trivial. Note that (4.1) reduces to the classical one for $sl(r+1|s+1)$; $r\neq s$ [BB2], if we drop the u dependence.

In this paper, we mainly consider the Bethe ansatz equations for a distinguished root system. The case for a non-distinguished root system will be achieved by some modifications of the set J_+, J_- and the function $z(a;u)$ without changing the set J and tableau sum rule (see appendices C, D). It will be interesting to extend a similar analysis presented in this paper for other Lie superalgebras, such as $\mathrm{osp}(m|2n)$.

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Appendix A. Example of the L operator and transfer matrix

In this section, we define the transfer matrix along the same line presented in [EK]. Let $L(u)^{ab}_{\alpha\beta}$ be the L operator [KulSk, PS1, PS2, Sc, BS] such that

$$L(u)^{aa}_{aa}=[u+2(-1)^{p(a)}] \qquad L(u)^{bb}_{aa}=[u],\ L(u)^{ba}_{ab}=[2(-1)^{p(a)p(b)}]q^{\mathrm{sign}(a-b)u} \qquad (A.1)$$

where we assume $a\neq b$; $a,b\in J$. The monodromy matrix $\mathcal{J}(u)$ is defined as

$$\mathcal{J}(u)^{a,\gamma_1\ldots\gamma_N}_{b,\beta_1\ldots\beta_N}=\sum_{a_1,\ldots,a_N}L(u)^{aa_N}_{\gamma_N\beta_N}L(u)^{a_Na_{N-1}}_{\gamma_{N-1}\beta_{N-1}}\cdots L(u)^{a_2a_1}_{\gamma_2\beta_2}L(u)^{a_1b}_{\gamma_1\beta_1}$$
$$\times(-1)^{\sum_{i=2}^{N}(p(\gamma_i)+p(\beta_i))\sum_{j=1}^{i-1}p(\gamma_j)}. \qquad (A.2)$$

The transfer matrix is defined as supertrace of the monodromy matrix

$$t(u)^{\gamma_1\ldots\gamma_N}_{\beta_1\ldots\beta_N} = \sum_{a=1}^{r+s+2} (-1)^{p(a)} \mathcal{J}(u)^{a,\gamma_1\ldots\gamma_N}_{a,\beta_1\ldots\beta_N}. \tag{A.3}$$

Thanks to the intertwining relation, the commutativity relation $[t(u), t(v)] = 0$ follows. The function $\mathcal{T}^1(u)$ defined in (2.13*a*) will coincide with the the spectrum of the transfer matrix $t(u)$ under the Bethe ansatz equation (2.6*a*) for relevant N_j. For example, for $r = 0, s = 1$, $N - N_1, N_1 - N_2, N_2$ (see (2.6*b*)) denote the number of γ_i equal to 1, 2, 3 in the set $\{\gamma_1, \ldots, \gamma_N\}$ respectively. Moreover, the function

$$\mathcal{T}^1(u) = \boxed{1} - \boxed{2} - \boxed{3} \tag{A.4}$$

coincides with Sutherland's solution [Su] on supersymmetric $t - J$ model presented in [EK] in the limit $q \to 1$, except for the overall scalar factor, after some redefinition.

Appendix B. On the expressions related to contravariant representations

In the main text, we have treated mainly the expressions related to covariant representations. For contravariant representations, we can also play a similar game. We often mark the expression related to contravariant representation with a dot. In the contravariant case, the relations (2.7)–(2.10) and (2.27) become respectively as follows:

$$\dot{J} = \{-1, -2, \ldots, -r - s - 2\} \qquad \dot{J}_+ = \{-1, -2, \ldots, -r - 1\}$$

$$\dot{J}_- = \{-r - 2, -r - 3, \ldots, -r - s - 2\} \tag{B.1}$$

$$-r - s - 2 \prec -r - s - 1 \prec \cdots \prec -1 \tag{B.2}$$

$$p(a) = \begin{cases} 0 & \text{for } a \in \dot{J}_+ \\ 1 & \text{for } a \in \dot{J}_- \end{cases} \tag{B.3}$$

$$\dot{z}(a; u) = \psi_a(u) \frac{Q_{-a-1}(u + r - s + a - 1) Q_{-a}(u + r - s + a + 2)}{Q_{-a-1}(u + r - s + a + 1) Q_{-a}(u + r - s + a)} \qquad \text{for } a \in \dot{J}_+$$

$$\dot{z}(a; u) = \psi_a(u) \frac{Q_{-a-1}(u - r - s - a - 1) Q_{-a}(u - r - s - a - 4)}{Q_{-a-1}(u - r - s - a - 3) Q_{-a}(u - r - s - a - 2)} \qquad \text{for } a \in \dot{J}_- \tag{B.4}$$

$$a_{r+1-j} = \xi_j - \xi_{j+1} \qquad \text{for } 1 \leqslant j \leqslant r$$

$$a_{r+1} = -\xi_1 - \dot{\mu}'_{s+1}$$

$$a_{r+s+2-j} = \dot{\mu}'_j - \dot{\mu}'_{j+1} \qquad \text{for } 1 \leqslant j \leqslant s \tag{B.5}$$

where $\xi_j = \max\{\dot{\mu}_j - s - 1, 0\}$; $\dot{\mu}'_{s+2} \leqslant r + 1$.

The functions (2.6*d*) and (2.11) take the form

$$P_a^{(j)}(u) = [u - w_j]^{\delta_{a,r+s+1}} \qquad \psi_a(u) = \begin{cases} P_{r+s+1}(u - 2) & \text{for } a = -r - s - 2 \\ P_{r+s+1}(u) & \text{for } a \in \dot{J} - \{-r - s - 2\} \end{cases} \tag{B.6}$$

if the quantum space is labelled by the contravariant Young superdiagram with shape $\dot{\tilde{\mu}} = (1^1)$;

$$P_a^{(j)}(u) = [u - w_j]^{\delta_{a,1}} \qquad \psi_a(u) = \begin{cases} P_1(u + r - s - 2) & \text{for } a = -1 \\ P_1(u + r - s) & \text{for } a \in \dot{J} - \{-1\} \end{cases} \tag{B.7}$$

if the quantum space is labelled by the covariant Young superdiagram with shape $\tilde{\mu} = (1^1)$.

If the quantum space is labelled by the contravariant Young superdiagram, in contrast to the covariant case, the parameter t_{r+1} on the left-hand side of the Bethe ansatz equation (2.6*a*) will be -1, since the $(r+1)$th Kac–Dynkin label takes negative values for the contravariant Young superdiagram [BMR]. For $-a \in \dot{J}$ and (B.4) with (B.7), the following relation holds

$$z(a;u) = (-1)^N \dot{z}(-a; s-r-u)|_{u_k^{(a)} \to -u_k^{(a)}, w_i \to -w_i}. \tag{B.8}$$

Note that this relation reduces to the crossing symmetry [R2] for sl_{r+1}, if we set $s = -1$ (see, also [KS1]). Pole freeness of the function $\dot{\mathcal{T}}_{\dot{\lambda}\subset\dot{\mu}}(u)$ under the Bethe ansatz equation (2.6*a*) can be proved in the same way as theorem 2.1.

Appendix C. Example of the non-distinguished simple roots case: $p(1) = 1, p(2) = 0, p(3) = 1$ grading

Let α_1 and α_2 be the simple roots of $sl(1|2)$ normalized so that $(\alpha_1|\alpha_1) = (\alpha_2|\alpha_2) = 0$ and $(\alpha_1|\alpha_2) = (\alpha_2|\alpha_1) = -1$ (see figure C1).

In this case, the sets (2.7) and (B.1) become $J_+ = \{2\}$, $J_- = \{1,3\}$; $\dot{J}_+ = \{-2\}$, $\dot{J}_- = \{-1,-3\}$. The function $z(a;u) = \boxed{a}_u$ $(a \in J)$ has the form

$$\boxed{1} = [u-2]^N \frac{Q_1(u+1)}{Q_1(u-1)} \qquad \boxed{2} = [u]^N \frac{Q_1(u+1)Q_2(u-2)}{Q_1(u-1)Q_2(u)}$$
$$\boxed{3} = [u]^N \frac{Q_2(u-2)}{Q_2(u)} \tag{C.1}$$

and the function $\dot{z}(a;u) = \boxed{a}_u$ $(a \in \dot{J})$ has the form

$$\boxed{-3} = [u-2]^N \frac{Q_2(u+1)}{Q_2(u-1)} \qquad \boxed{-2} = [u]^N \frac{Q_1(u-2)Q_2(u+1)}{Q_1(u)Q_2(u-1)}$$
$$\boxed{-1} = [u]^N \frac{Q_1(u-2)}{Q_1(u)}. \tag{C.2}$$

Here we assume the quantum spaces are labelled by Young superdiagrams with shapes $\tilde{\mu} = (1^1)$ and $\dot{\tilde{\mu}} = (1^1)$ respectively; for simplicity, inhomogeneity parameters w_i are set to 0. For example, for $\lambda = \phi$; $\mu = (2^1)$, (2.15) has the form

$$\begin{aligned}
\mathcal{T}_2^1(u) = &-\boxed{1}\boxed{2} + \boxed{1}\boxed{3} + \boxed{2}\boxed{2} - \boxed{2}\boxed{3} \\
= &-[u-3]^N[u+1]^N \frac{Q_1(u+2)Q_2(u-1)}{Q_1(u-2)Q_2(u+1)} \\
&+[u-3]^N[u+1]^N \frac{Q_1(u)Q_2(u-1)}{Q_1(u-2)Q_2(u+1)} \\
&+[u-1]^N[u+1]^N \frac{Q_1(u+2)Q_2(u-3)}{Q_1(u-2)Q_2(u+1)} \\
&-[u-1]^N[u+1]^N \frac{Q_1(u)Q_2(u-3)}{Q_1(u-2)Q_2(u+1)}
\end{aligned} \tag{C.3}$$

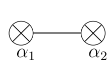

Figure C1. Dynkin diagram for the Lie superalgebra $sl(1|2)$ corresponding to the non-distinguished simple roots: $\deg(\alpha_1) = \deg(\alpha_2) = 1$.

and for $\lambda = \phi; \mu = (1^2)$, (2.15) has the form

$$\begin{aligned}
\mathcal{T}_1^2(u) &= \frac{1}{[u-1]^N}\left(\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} - \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} - \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}\right) \\
&= [u-3]^N \frac{Q_1(u+2)}{Q_1(u-2)} - [u-1]^N \frac{Q_1(u+2)Q_2(u-3)}{Q_1(u-2)Q_2(u-1)} \\
&\quad + [u-1]^N \frac{Q_1(u+2)Q_2(u-3)}{Q_1(u)Q_2(u-1)} - [u+1]^N \frac{Q_1(u+2)Q_2(u-3)}{Q_1(u)Q_2(u+1)} \\
&\quad + [u+1]^N \frac{Q_2(u-3)}{Q_2(u+1)}.
\end{aligned} \tag{C.4}$$

We note that the function $\dot{\mathcal{T}}^1(u)$ associated with the contravariant Young superdiagram $\dot{\mu} = \phi; \dot{\lambda} = (1^1)$:

$$\dot{\mathcal{T}}^1(u) = -\boxed{-3} + \boxed{-2} - \boxed{-1} \tag{C.5}$$

coincides with Essler and Korepin's solution [EK] on the supersymmetric $t - J$ model in the limit $q \to 1$ except overall scalar factor after some redefinition†. Pole freeness of the functions $\mathcal{T}^a(u)$ and $\dot{\mathcal{T}}^a(u)$ under the Bethe ansatz equation (2.6*a*) can be proved in the same way as theorem 2.1.

Appendix D. Example of non-distinguished simple roots case: $p(1) = p(2) = 1, p(3) = 0$ grading

Let α_1 and α_2 be the simple roots of $sl(1|2)$ normalized so that $(\alpha_1|\alpha_1) = -2, (\alpha_2|\alpha_2) = 0$ and $(\alpha_1|\alpha_2) = (\alpha_2|\alpha_1) = 1$ (see figure D1). In this case, the sets (2.7) and (B.1) become $J_+ = \{3\}$, $J_- = \{1, 2\}$; $\dot{J}_+ = \{-3\}$, $\dot{J}_- = \{-1, -2\}$. The function $z(a; u) = \boxed{a}_u$ $(a \in J)$ has the form

$$\begin{aligned}
&\boxed{1} = [u-2]^N \frac{Q_1(u+1)}{Q_1(u-1)} \qquad \boxed{2} = [u]^N \frac{Q_1(u-3)Q_2(u)}{Q_1(u-1)Q_2(u-2)} \\
&\boxed{3} = [u]^N \frac{Q_2(u)}{Q_2(u-2)}
\end{aligned} \tag{D.1}$$

and the function $\dot{z}(a; u) = \boxed{a}_u$ $(a \in \dot{J})$ has the form

$$\begin{aligned}
&\boxed{-3} = [u+2]^N \frac{Q_2(u-1)}{Q_2(u+1)} \qquad \boxed{-2} = [u]^N \frac{Q_1(u+2)Q_2(u-1)}{Q_1(u)Q_2(u+1)} \\
&\boxed{-1} = [u]^N \frac{Q_1(u-2)}{Q_1(u)}.
\end{aligned} \tag{D.2}$$

Here we assume the quantum spaces are labelled by Young superdiagrams with shapes $\tilde{\mu} = (1^1)$ and $\tilde{\dot{\mu}} = (1^1)$ respectively; for simplicity, inhomogeneity parameters w_i are set to 0.

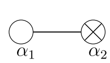

Figure D1. Dynkin diagram for the Lie superalgebra $sl(1|2)$ corresponding to the non-distinguished simple roots: $\deg(\alpha_1) = 0, \deg(\alpha_2) = 1$.

† This coincidence does not necessarily mean the coincidence of underlying representation of Lie superalgebra $sl(r+1|s+1)$.

For example, for $\lambda = \phi$; $\mu = (2^1)$, (2.15) has the form

$$\begin{aligned}
\mathcal{T}^1_2(u) &= \boxed{1\,|\,2} - \boxed{1\,|\,3} - \boxed{2\,|\,3} + \boxed{3\,|\,3} \\
&= [u-3]^N[u+1]^N \frac{Q_2(u+1)}{Q_2(u-1)} - [u-3]^N[u+1]^N \frac{Q_1(u)Q_2(u+1)}{Q_1(u-2)Q_2(u-1)} \\
&\quad -[u-1]^N[u+1]^N \frac{Q_1(u-4)Q_2(u+1)}{Q_1(u-2)Q_2(u-3)} \\
&\quad +[u-1]^N[u+1]^N \frac{Q_2(u+1)}{Q_2(u-3)}
\end{aligned} \tag{D.3}$$

and for $\lambda = \phi$; $\mu = (1^2)$, (2.15) has the form

$$\begin{aligned}
\mathcal{T}^2_1(u) &= \frac{1}{[u-1]^N}\left(\boxed{\begin{matrix}1\\1\end{matrix}} + \boxed{\begin{matrix}1\\2\end{matrix}} - \boxed{\begin{matrix}1\\3\end{matrix}} + \boxed{\begin{matrix}2\\2\end{matrix}} - \boxed{\begin{matrix}2\\3\end{matrix}} \right) \\
&= [u-3]^N \frac{Q_1(u+2)}{Q_1(u-2)} + [u-1]^N \frac{Q_1(u-4)Q_1(u+2)Q_2(u-1)}{Q_1(u-2)Q_1(u)Q_2(u-3)} \\
&\quad -[u-1]^N \frac{Q_1(u+2)Q_2(u-1)}{Q_1(u)Q_2(u-3)} + [u+1]^N \frac{Q_1(u-4)Q_2(u+1)}{Q_1(u)Q_2(u-3)} \\
&\quad -[u+1]^N \frac{Q_1(u-2)Q_2(u+1)}{Q_1(u)Q_2(u-3)}.
\end{aligned} \tag{D.4}$$

We note that the function $\dot{\mathcal{T}}^1(u)$ associated with the contravariant Young superdiagram with shape $\dot{\lambda} = \phi$; $\dot{\mu} = (1^1)$

$$\dot{\mathcal{T}}^1(u) = \boxed{-3} - \boxed{-2} - \boxed{-1} \tag{D.5}$$

coincides with Lai's solution [L] on supersymmetric $t - J$ model presented in [EK] in the limit $q \to 1$ except overall scalar factor after some redefinition†. Pole freeness of the functions $\mathcal{T}^a(u)$ and $\dot{\mathcal{T}}^a(u)$ under the Bethe ansatz equation (2.6*a*) can be proved in the same way as theorem 2.1.

Appendix E. Other representation of $\mathcal{T}^a$ and $\mathcal{T}_m$

For simplicity, we assume the vacuum part is formally trivial. Define the functions $\mathcal{A}^a$, $\mathcal{B}^a$, $\mathcal{A}_m$ and $\mathcal{B}_m$ by the generating series such that

$$\sum_{k=-\infty}^{\infty} \mathcal{A}_k(u+k-1)X^k = (1 - z(1;u)X)^{-1} \cdots (1 - z(r+1;u)X)^{-1} \tag{E.1}$$

$$\sum_{l=-\infty}^{\infty} \mathcal{B}^l(u+l-1)X^l = (1 - z(r+2;u)X) \cdots (1 - z(r+s+2;u)X) \tag{E.2}$$

$$\sum_{k=-\infty}^{\infty} \mathcal{B}_k(u+k-1)X^k = (1 + z(r+s+2;u)X)^{-1} \cdots (1 + z(r+2;u)X)^{-1} \tag{E.3}$$

$$\sum_{l=-\infty}^{\infty} \mathcal{A}^l(u+l-1)X^l = (1 + z(r+1;u)X) \cdots (1 + z(1;u)X). \tag{E.4}$$

† This coincidence does not necessarily mean the coincidence of underlying representation of Lie superalgebra $sl(r+1|s+1)$.

Combining these relations, we obtain

$$\mathcal{T}^a(u) = \sum_{l=0}^{\min(r+1,a)} \mathcal{B}_{a-l}(u-l)\mathcal{A}^l(u+a-l) \tag{E.5}$$

$$\mathcal{T}_m(u) = \sum_{l=0}^{\min(s+1,m)} \mathcal{A}_{m-l}(u-l)\mathcal{B}^l(u+m-l). \tag{E.6}$$

Note that these functions $\mathcal{A}_m(u)$ and $\mathcal{A}^a(u)$ are analogous to eigenvalue formulae of transfer matrices in dressed-vacuum form of fusion $U_q(sl^{(1)}_{r+1})$ vertex model labelled by Young diagrams with shapes (m^1) and (1^a) respectively. We also note that the functions $\mathcal{B}^a(u)$ and $\mathcal{B}_m(u)$ are analogous to eigenvalue formulae of transfer matrices in dressed-vacuum form of fusion $U_q(sl^{(1)}_{s+1})$ vertex model labelled by Young diagrams with shapes (1^a) and (m^1) respectively.

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